

Conformal general relativity contains the quantum

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Abstract

Based on the de Broglie-Bohm relativistic quantum theory of motion we show that the conformal formulation of general relativity, being linked with a Weyl-integrable geometry, may implicitly contain the quantum effects of matter. In this context the Mach's principle is discussed.

In recent papers [1,2] the singularity problem has been treated from a point of view that is based upon the conformal transformation technique. Under a conformal transformation of the spacetime metric

$$\hat{g}_{ab} = \Omega^2(x)g_{ab}, \quad (1)$$

where $\Omega(x)$ is a non-vanishing smooth function, the spacetime coincidences [coordinates] are unchanged. Correspondingly, the experimental observations [measurements], being nothing but just verifications of these coincidences are unchanged too by this conformal transformation.

Dicke showed that the transformation (1) can be interpreted as a point-dependent transformation of the units of length, time and mass [3]. In fact, if one takes the arc-length

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as one's unit of measurement of length and time [the speed of light $c = 1$] hence, since $d\hat{s} = \Omega(x)ds$, the unit of length in the conformal frame is point-dependent even if ds is a constant along a geodesic in the original frame. In the same way, in Brans-Dicke (BD) theory [4] the unit of mass transforms under (1) as $\hat{m} = \Omega(x)^{-1}m$. Hence, even if the unit of mass in the original frame m is a constant, in the conformal frame \hat{m} will be point-dependent.

In Ref. [2] the requirement that the laws of physics must be invariant not only under general coordinate transformations but, also, under point-dependent transformations of the units of length, time and mass, was raised to a category of a postulate and was used there as a selection principle while studying different generic effective theories of gravity. It has been shown there that the only surviving candidate for a final low-energy theory of spacetime [among those studied therein] is the one with non-minimal coupling of the dilaton both to curvature and to the Lagrangian of the matter fields. The requirement of invariance under point-dependent transformations of units leads one to consider geometries that admit units of measure that may vary along transport [2]. These geometries are called generically as Weyl geometries [5].

Usually the effective theories with minimal coupling of the matter fields, i. e., with the matter part of the action of the kind

$$S_{matter} = 16\pi \int d^4x \sqrt{-g} L_{matter}, \quad (2)$$

are linked with manifolds of Riemannian structure since, in this case, the [time-like] matter particles follow free-motion paths that are solutions of the differential equation

$$\frac{d^2x^a}{ds^2} + \{^a_{mn}\} \frac{dx^m}{ds} \frac{dx^n}{ds} = 0, \quad (3)$$

where $\{^a_{bc}\} \equiv \frac{1}{2}g^{an}(g_{bn,c} + g_{cn,b} - g_{bc,n})$ are the Christoffel symbols of the metric g_{ab} . I. e., these coincide with the Riemannian geodesics of the metric. In general a Riemann configuration is characterized by the requirement that the covariant derivatives of the metric tensor vanish, i.e.,

$$g_{ab;c} = 0, \quad (4)$$

where semicolon denotes covariant differentiation in a general affine sense.¹ Fulfillment of this condition leads the manifold affine connections Γ_{bc}^a to become identical to the Christoffel symbols $\{^a_{bc}\}$ of the metric (i.e., $\Gamma_{bc}^a \equiv \{^a_{bc}\}$). Hence, a Riemann configuration of spacetime is characterized by the requirement that $g_{ab||c} = 0$, where the double bar denotes covariant differentiation defined through the Christoffel symbols of the metric [instead of the affine connections Γ_{bc}^a]. This requirement implies that vector lengths do not change under parallel transport, meaning that the units of measure of the geometry are point-independent. In particular, general relativity with an extra scalar [dilaton] field with the canonical action

$$S = \int d^4x \sqrt{-g} (R - \alpha (\nabla\psi)^2 + 16\pi L_{matter}), \quad (5)$$

where R is the Ricci scalar in terms of the metric g_{ab} , ψ is the dilaton field, and α is a free [constant] parameter, is naturally linked with Riemann geometry with point-independent units of measure. It should be noted that, when we put in Eq. (5) $\alpha = 0$ [ψ arbitrary] or when $\psi = \text{const}$, we recover usual Einstein's formulation of general relativity.

Under the conformal rescaling (1) with $\Omega^2 = e^{-\psi}$, the action (5) is mapped into its conformal action

$$S = \int d^4x \sqrt{-\hat{g}} (\hat{\phi} \hat{R} - \frac{\alpha - \frac{3}{2}}{\hat{\phi}} (\hat{\nabla} \hat{\phi})^2 + 16\pi \hat{\phi}^2 L_{matter}), \quad (6)$$

where \hat{R} is the curvature scalar in terms of the metric \hat{g}_{ab} , $\hat{\phi}$ is the transformed dilaton [the following replacement has been used $\hat{\phi} = e^\psi$], and the matter fields are now non-minimally coupled to the dilaton. At the same time, under this transformation of the spacetime metric, manifolds of Riemann structure are mapped onto manifolds of conformally-Riemannian

¹We use, mainly, the notation of Ref. [6]

structure. Therefore, in the theory that is derivable from the action Eq. (6) (Conformal general relativity) the underlying manifolds are of conformally-Riemannian nature. Conformally-Riemannian manifolds are also acknowledged as Weyl-integrable spaces [6].² In effect, under the rescaling (1) with $\Omega^2 = e^{-\psi}$, the Riemannian requirement Eq. (4) is transformed into the following requirement

$$\hat{g}_{ab;c} = \psi_{,c} \hat{g}_{ab}, \quad (7)$$

where now semicolon denotes covariant differentiation in a general affine sense with $\hat{\Gamma}_{bc}^a$ being the affine connection of the conformal manifold. It is given through the Christoffel symbols of the metric with a hat $\{\hat{g}_{bc}^a\}_{hat}$ and the derivatives of the scalar function [dilaton] ψ as

$$\hat{\Gamma}_{bc}^a \equiv \{\hat{g}_{bc}^a\}_{hat} - \frac{1}{2}(\psi_{,b} \delta_c^a + \psi_{,c} \delta_b^a - \hat{g}_{bc} \hat{g}^{an} \psi_{,n}). \quad (8)$$

If one compares Eq. (7) with the requirement of nonvanishing covariant derivative of the metric tensor \hat{g}_{ab} in the most generic cases of Weyl geometries [6]:

$$\hat{g}_{ab;c} = f_c \hat{g}_{ab}, \quad (9)$$

in which $f_a(x)$ is the Weyl gauge vector, one arrives at the conclusion that the conformally-Riemannian geometry -characterized by Eq. (7)- is a particular type of Weyl geometry in which the gauge vector is the gradient of a scalar function ψ [the dilaton]. This particular type of Weyl spaces is called a Weyl-integrable spacetime (WIST) since length variations are integrable along closed paths: $\oint dl = 0$, where $dl = l dx^n \psi_{,n}$, and $l \equiv \hat{g}_{nm} V^n V^m$ is the length of the vector $V^a(x)$ being parallelly transported along of the closed path. For this reason, in manifolds of WIST configuration, the disagreement with observations due to the "second clock effect" [7] -that is inherent to Weyl spacetimes in general- is overcome [6].

²These represent a special case of Weyl geometry that is free of the "second clock effect". This effect is in disagreement with observations [7].

Therefore, under the transformation (1) with $\Omega^2 = e^{-\psi}$ Riemann geometry is mapped into a WIST geometry. In other words, in theories showing non-minimal coupling of the matter fields to the metric, in particular with the matter part of the action of the kind

$$S_{matter} = 16\pi \int d^4x \sqrt{-\hat{g}} e^{2\psi} L_{matter} = 16\pi \int d^4x \sqrt{-\hat{g}} \hat{\phi}^2 L_{matter} \quad (10)$$

[it is conformal to (2)], the nature of the underlying manifold is that of a WIST configuration. The equations of free-motion of a material test particle that are derivable from Eq. (10)

$$\frac{d^2 x^a}{d\hat{s}^2} + \{\overset{a}{mn}\}_{hat} \frac{dx^m}{d\hat{s}} \frac{dx^n}{d\hat{s}} - \frac{\psi_{,n}}{2} \left(\frac{dx^n}{d\hat{s}} \frac{dx^a}{d\hat{s}} - \hat{g}^{na} \right) = 0, \quad (11)$$

are conformal to Eq. (3). Equations (11) coincide with the equations defining geodesic curves in spacetimes of WIST configuration. These can also be obtained with the help of the variational principle $\delta \int e^{-\frac{\psi}{2}} d\hat{s} = 0$, that is conformal to $\delta \int ds = 0$.

The requirement of nonvanishing covariant derivative of the metric tensor \hat{g}_{ab} [Eq. (7)] implies that vector lengths may vary along transport or, in other words, that the units of measure may change locally. Therefore WIST geometry represents a generalization of Riemann geometry to include units of measure with point-dependent length.

In other words, under the conformal rescaling (1) with $\Omega^2 = e^{-\psi}$ theories with minimal coupling of the matter fields to the metric [for instance those with the matter part of the action of the kind (2)] being uniquely linked with manifolds of Riemann structure, are mapped into theories with non-minimal coupling of the matter fields to the metric [with the matter part of the action of the kind (10)] in which the nature of the underlying manifold is of WIST structure. In particular, the Riemannian geodesics of the metric g_{ab} are mapped under this transformation onto geodesics in manifolds of a WIST structure that are specified by the conformal metric \hat{g}_{ab} and the gauge vector $\psi_{,a}$ [or, alternatively, $\hat{\phi}_{,a}/\hat{\phi}$]. This means that under (1) the metric g_{ab} that can be measured with the help of clocks and rods made of ordinary matter in manifolds of Riemann geometry is mapped into the metric \hat{g}_{ab} that can be measured with the help of clocks and rods made of ordinary matter in a manifold

of WIST structure. It should be remarked, however, that the metric without a hat and the one with it are not physically equivalent.³ In fact, it can be checked that under the transformations of the one-parameter group of conformal transformations [8]

$$\hat{g}_{ab} = \hat{\phi}^{-\sigma} g_{ab}, \quad \hat{\phi} = \phi^{1-\sigma}, \quad \hat{\alpha} = \frac{\alpha}{(1-\sigma)^2}, \quad (12)$$

$\sigma \neq 1$, the action (6) is invariant in form while the action (5) is not invariant under the transformations of this group. In Ref. [2] it has been argued that this group of transformations can be identified with the one-parameter group of point-dependent transformations of the units of length, time, and mass.⁴ Hence, the action (6) has a higher degree of symmetry than its conformal action (5): It is invariant under point-dependent transformations of the units of length, time, and mass.

In this letter we shall study an idea that is based on the de Broglie-Bohm quantum theory of motion [first presented in reference [9]] within the frame of WIST geometry being naturally linked with the conformal formulation of general relativity [action(6)]. In this context we shall briefly discuss on the Mach's principle and we shall hint at its possible connection with the quantum.

Recently it was shown [9] that, within the context of the de Broglie-Bohm relativistic quantum theory [10], the quantum effects of matter can be explicitly included in the metric through a conformal transformation of the metric. In fact, under a conformal transformation of the kind (1), the geodesic equations of Riemann geometry [Eq.(3)] are mapped into the Eq.(11) defining a non-geodesic motion on a spacetime with metric \hat{g}_{ab} provided that the Riemaniann structure of the geometry is preserved under (1).

If we set $\hat{\phi} = 1 + Q$, where Q is the matter quantum potential [10], hence the last term

³Both metrics are however observationally equivalent

⁴The transformation (1.1) with $\Omega^2 = e^{-\psi} = \hat{\phi}^{-1}$ does not belong to this group since it is the particular case when in Eq. (12) $\sigma = 1$

[third term] in the left hand side (LHS) of Eq.(11) represents the quantum force [9]. This quantum force causes a deviation of the motion of a particle from being geodesic [in terms the metric field g_{ab}] to being a non-geodesic one [in terms the metric field \hat{g}_{ab}]. We recall that a basic assumption here is that the Riemannian nature of the geometry is preserved under (1). We are not concerned in this letter with the validity of the ideas developed in Ref. [9]. Our goal here is to study the consequences of taking an action of the kind (6) for the description of the laws of gravity [the underlying manifold is of WIST structure] when the de Broglie-Bohm quantum theory of motion is approached. In this context, following Ref. [9], we shall set $\hat{\phi} \equiv 1 + Q$, where, as before, Q is the matter quantum potential.

The basic consideration in this letter is that, under (1), spacetimes of Riemannian structure are mapped onto Weyl-integrable spacetimes. Correspondingly, the equation (3) defining geodesic curves in Riemann geometry is mapped into the equation (11) defining geodesic curves in spacetimes of WIST configuration. This means that a free-falling test particle would not 'feel' the quantum force if the motion is interpreted on the grounds of a WIST geometry and, correspondingly, the metric with a hat \hat{g}_{ab} is taken to give the physical interpretation of the experimental observations. This hints at the conclusion that Weyl-integrable geometry contains implicitly the quantum effects of matter. Consequently the conformal formulation of general relativity [action (6)] -being naturally linked with spacetimes of WIST structure- is already a theory of gravity implicitly containing the quantum effects. Hence, we feel, it is not causal that under (1) incomplete geodesics over Riemann geometry can be mapped [in principle] into complete geodesics over a WIST geometry [1,2]. This last geometry is compatible with singularity free spacetimes even if its conformal [Riemann] geometry is linked with spacetimes containing singularities. We recall that the inevitability of spacetime singularities in spacetimes of Riemann geometry is usually linked with the lack of quantum considerations in the canonical formulation of general relativity [action (5)].

The non-local character of the matter quantum potential Q allows one to consider in a natural way the validity of the Mach's principle within the context of conformal general relativity and, correspondingly, within the context of a WIST geometry. Under the rescaling

(1) the inertial mass of a given elementary particle transforms like [1,2]

$$\hat{m} = e^{-\frac{1}{2}\psi} m \quad (13)$$

where m is the constant mass of the particle in canonical general relativity. Correspondingly \hat{m} is the variable inertial mass of the particle within the context of conformal general relativity. We recall that the geometrical interpretation of the experimental observations is to be given on spacetimes of WIST structure [with the law (7) of length transport]. It is very encouraging that the mass of a single particle, being local in nature, is determined by a global parameter such as the curvature k that characterizes the entire universe. We shall show this with an example. For simplicity let us take an homogeneous and isotropic dust-filled Friedmann-Robertson-Walker (FRW) universe with the Riemannian line-element

$$ds^2 = -dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (14)$$

where a is the time-dependent Riemannian scale factor, $k = 0, \pm 1$ is the spatial curvature and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. The field equations derivable from the action (5) for canonical general relativity are the following.

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}\mu + \frac{\alpha}{6}\psi^2, \quad \ddot{\psi} = -3\frac{\dot{a}}{a}\dot{\psi}, \quad \dot{\mu} + 3\frac{\dot{a}}{a}\mu = 0, \quad (15)$$

where the overdot means derivative with respect to the Riemannian cosmic time t . The last equation in (15) can be integrated to give $\mu = \frac{3}{8\pi} \frac{M}{a^3}$ [M is an integration constant].

After integrating once the second equation in (15) one obtains

$$\dot{\psi} = \pm \frac{N}{a^3}, \quad (16)$$

where N is another integration constant and the "+" and "-" signs define two possible branches of our solution. For further simplification of the analysis we shall consider the

particular case with $\alpha = 0$ in the first equation in (15).⁵ Hence, after the change of time variable $dt = \frac{a^2}{\sqrt{M}}d\eta$ this last equation can be readily integrated to give

$$a = \frac{4}{\eta^2 + \frac{4k}{M}}. \quad (17)$$

Consequently integration of Eq.(16) leads to the following result

$$\psi^\pm = \psi_0 \pm \frac{N}{12\sqrt{M}}\eta(\eta^2 + \frac{12k}{M}), \quad (18)$$

where ψ_0 is yet another integration constant. Without loss of generality we can take it to be zero. Hence, if we substitute (18) in Eq.(13) we see that the inertial mass of an elementary particle [being a local feature of matter] is affected by the integration constants -given through the boundary conditions- and by the curvature of the universe (k). If we take into account the Eq.(18) then the matter quantum potential can be implicitly written in terms of the integration constants and the curvature k through [$e^\psi = 1 + Q$]

$$Q^\pm = \exp[\pm \frac{N}{12\sqrt{M}}\eta(\eta^2 + \frac{12k}{M})] - 1. \quad (19)$$

Take, for simplicity, a flat FRW universe [$k = 0$]. In this case the time coordinate η takes values in the interval $0 \leq \eta \leq \infty$ [the case $-\infty \leq \eta \leq 0$ represents the time-reversed situation]. Hence, since the Newton's constant is proportional to $e^{-\psi}$, we shall choose the "- branch of the solution (18) as the "physical" branch. In fact, in this branch of the solution the Newton's constant evolves from being the unity at $\eta = 0$ [the scale factor a is infinite] to being infinite in the infinite future $\eta = +\infty$ [the Riemannian scale factor a is zero meaning that, within the context of Riemann geometry, the universe collapses into a final singularity] as it should be.

As we have properly remarked, the physical interpretation of our results is to be given in terms of the magnitudes with a hat and, correspondingly within the context of a Weyl-

⁵This value for α is supported by current experiments confirming Einstein's general relativity

integrable geometry. In the "-" branch of the solution, the conformal cosmic time \hat{t} is given in terms of the time coordinate η through the following approximate relationship $\hat{t} \sim \int \frac{e^{3\eta}}{\eta^4} d\eta$ while the conformal scale factor $\hat{a} \sim \frac{e^{3\eta}}{\eta^2}$. It can be verified that, in this case, the flat FRW universe, when interpreted within the context of a WIST geometry [magnitudes with a hat], is free of the cosmological singularity [1,2]. It evolves from being infinite in the infinite past [$\hat{t} = -\infty$] through a bounce at a minimum value of the scale factor \hat{a} to being infinite in the infinite future [$\hat{t} = +\infty$]. This bouncing universe is regular everywhere. The matter quantum potential monotonically decreases from $Q = 0$ at the infinite past to $Q = -1$ at the infinite future. Hence, since the absolute value of the quantum potential increases with the evolution of the universe, this means that, in this particular case, the contribution from the quantum increases with the evolution of the universe. I. e., the universe evolves from being classical in the infinite past to being quantum in nature in the infinite future. Just the opposite situation is obtained if in the untransformed [Riemannian] description the universe evolves from a global cosmological singularity at some initial time [the time-reversed situation].

Summing up. If we study the de Broglie-Bohm approach to the quantum description of the world within the context of the conformal formulation of general relativity [that is naturally linked with a Weyl-integrable geometry], two main conclusions arise. First, the quantum effects of matter are implicitly contained in the Lagrangian formulation of the theory through the dilaton field. In other words, the quantum effects are already implicit in the laws of the Weyl-integrable geometry. The second conclusion is linked with the fact that the Mach's principle, being quantum in nature, acts at the scale of the entire universe, i.e., it acts like a bridge between the micro and the macro scales.

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